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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Raymond Garver

Raymond Garver, Associate Professor of Mathematics at the University of California at Los Angeles, died November 7, 1935, after a brief illness. Professor Garver was born in Iowa City, Iowa, on October 30, 1901. He received the A. B. degree at the University of Montana in 1922, the M. A. degree from the same University in 1924, and the Ph. D. degree from the University of Chicago in 1926. Before he joined the faculty of the University of California at Los Angeles in 1928, Dr. Garver spent two years as an instructor in mathematics at the University of Rochester.

Professor Garver was a young man of high promise in mathematics. He was a conscientious and inspiring teacher, loved and respected by his students and faculty colleagues. He was actively engaged in high quality research throughout the period of his professional life. His passing was a great loss to mathematics and to the University that he served.

Dr. Garver's published research* consists of approximately fifty-five articles. These articles appeared in *The Bulletin* of the American Mathematical Society, *The American Journal of Mathematics*, *Annals of Mathematics*, and numerous other American and foreign journals. His major interest was in the field of Algebraic Equations as is evidenced by the fact that forty-four of his papers were in this field. He also published four important papers on topics in Group Theory. The remainder of his work was concerned with Mathematical Problems in Economics. This latter field seemed to offer increasing attraction for him and it is quite probable that, had he continued to live, he would have directed a large portion of his energies in that field. A detailed listing of Dr. Garver's articles is not given here. Exact reference to most of these are to be found under "Annual List of Published Papers" in the *Bulletin* of the American Mathematical Society for the years 1927-1935, inclusive.

W. M. WHYBURN.
University of California at Los Angeles.

*A detailed list of his published papers, also prepared by Professor Whyburn, will appear in a later issue of the Magazine.—Editor.

Warren Colburn and His Influence on Arithmetic in the United States

By A. W. RICHESON
University of Maryland

Few, if any, of the writers on the history of American mathematics have made more than a passing mention of Warren Colburn. It seems without doubt that he initiated a movement which was destined to revolutionize the writing of text-books as well as the teaching of arithmetic in the United States.

Warren Colburn was born of poor parents on March 1, 1793 at Delham, Massachusetts.[1] In the early years of his life his parents moved from place to place in Massachusetts and the neighboring states in order to give him the best education available at that time. His elementary education consisted of attendance at the district summer school and later attendance at the district winter school. Along with his elementary schooling he was given every opportunity during his vacations to study machinery. In 1810 the Colburn family moved to Pawtucket, Rhode Island so that he might study machinery under John Fields a well known New England machinist.

At the age of 22 Colburn began to fit himself for college, in particular Harvard. As was the usual custom at that time he studied under a private tutor, a clergyman in the neighborhood. This preparation seems to have taken slightly over one year, as he entered Harvard and graduated after four years in 1820. After graduating he taught a private school of his own for two and one-half years. This seems to have been the extent of his teaching with the possible exception of some tutoring while he was a student at Harvard. He gave up teaching to accept a position in a manufacturing plant at Waltham, Massachusetts. He resigned this position in 1824 to become superintendent of the Lowell Manufacturing Plant at Lowell, Massachusetts, a position he held until his death on September 13, 1833.

Colburn's personal appearance was decidedly pleasing, and his disposition even and sweet. By nature he was rather studious and gave the impression of being abstract. At all times he was benevolent, intellectual, and refined. Although he lectured on many occas-

[1] See *Barnard's Journal*, Vol. II, pp. 294-305.

ions, he was a rather poor public speaker, a fact partly due, no doubt, to a slight hesitancy in his speech. One of his leading traits was his slowness of mind, and he gave long-continued thought to any problem which he attempted to solve. A result of this patience and slowness is shown in the solid foundations on which his conclusions were based.

Colburn's education was that of the average young man of the day entering and graduating from Harvard. A study of the courses in mathematics offered at Harvard about 1820 will indicate that arithmetic was studied during the freshman year along with algebra and geometry and that the course in mathematics extended over the entire four years.[2] It is evident that he mastered the calculus and read at least a large part of Laplace's great work. As a student Colburn showed a remarkable ability in mathematical analysis.

It is very probable that he gave considerable thought to the improvement of the existing system of education, especially in mathematics, while he was a student. Since his teaching career consisted of less than three years, he surely had at least worked out the general plan for the manuscript of his First Lessons in Intellectual Arithmetic. It is also certain that he began writing the manuscript as soon as he started teaching, and no doubt its purpose was for use in his own classes. He continued his interest in educational work long after he he stopped teaching and entered the business world. As evidence of of this fact we have only to look at the number and topics of the popular lectures he delivered in the following years. The purpose of these lectures was to present common and useful subjects in such a way as to gain attention, and in such a connection as to enlighten and enrich the popular mind with scientific subjects.[3] His aim was to bridge the gap between the college hall and the common school by means of these lectures. Some of the topics upon which he lectured were: history of animals, changes of seasons, light, electricity, thunder and lightning, and other natural phenomena.

Before discussing Colburn's works and influence on the teaching of elementary mathematics, it will be well to consider briefly the nature of published text-books and the methods of teaching arithmetic at the opening of the 19th century. The principal text-books published in this country at that time were by Pike, Adams, and Dilworth.[4] These texts were little more than a set of ill-formed rules, which were to be committed to memory, followed by sets of problems for solution.

[2]See Smith and Ginsburg, *A History of Mathematics in America Before* 1900, pp. 70-71; also Cajori, *The Teaching and History of Mathematics in the United States*, pp. 132-133.

[3]Barnard, ibid., p. 304.

[4]See Cajori, ibid, pp. 106-113; also Karpinski, ***The History of Arithmetic***, **pp. 78-99.**

There was no thought given to reasoning nor to problems for oral drill and practice. In many cases a dozen or more rules were given which might easily have been combined into one or two fundamental principles. As an example of this, rules were given in nearly all the texts of the day for computing interest with common fractions, and then a little further on a different set of rules was given for the same computations with decimal fractions. It is evident that the preparation of the teachers and the methods of teaching were in no way superior to the texts in use. This fact is substantiated by the following quotation from the key of the 1821 edition of the First Lessons in Intellectual Arithmetic by Colburn: "Instructors who may never have attended to fractions need not be afraid to undertake to teach this book. The author flatters himself that the principles are so illustrated, and the processes made so simple, that anyone, who, shall undertake to teach it, shall find himself familiar with fractions before he is aware of it, although he knew nothing of them before; and that everyone will acquire a facliity in solving questions, which he did not possess before." It should be borne in mind that the fractions referred to here were of the simplest type and that they were prepared for children from 5 to 8 years of age.

Instruction was given by the "ciphering" method; that is, there was practically no teaching or instruction whatever. Few of the children were provided with texts and as a result the teacher set the "sums", as they were called, and the child went back to his seat and attempted to solve the problem. After he had worked over the problem, it was again brought to the teacher; if the solution corresponded to the one in the copy-book, he was given another to work; if not, he was sent back to his seat to work on it further. Often it is questionable whether or not the teacher was able to work many of these problems.

The characteristics of the teaching and the text-books of the period were as follows: mental arithmetic was absent; cancellation was unknown; mensuration was of the English system rather than the French, proportion was taught as the "Rule of Three," and this rule did not suggest the equality of two ratios as the underlying principle of proportion; the texts of the day gave two methods of solving problems, one the method of "single position", the other the method of "double position". This deductive method of teaching along with the poor text-books produced, as would be expected, students who were problem solvers. In so far as the problems followed some rule that had been given was the student able to solve the problems. It is easily seen that the English influence was strongly felt in the writing

of text-books and the teaching of arithmetic until the close of the 18th century.

With the opening of the 19th century the French influence began to be felt in the United States. This was partly due to the bitterness felt, as a result of the Revolutionary War, by the educated people of the United States towards anything English. Along with this French influence came the effects of the principles of education as set down by the Swiss educational reformer, Pestalozzi. The first book on arithmetic to appear in the United States embodying the fundamental principles of Pestalozzi was the First Lessons in Intellectual Arithmetic by Warren Colburn. It is questionable whether or not Colburn had actually studied Pestalozzi in detail; however, it is quite certain that he had at least read, while at Harvard, an outline of Pestalozzi's works.

At this time there also arose a demand for class instruction, which in itself helped to bring about a change in the writing of text-books and in the methods of teaching. As the population of the country increased, the size of the classes naturally grew, and the demand for elementary education increased proportionally; this of course brought about the necessity for a radical change in the old methods of teaching the subject. These facts no doubt had their influence on Colburn in writing his texts.

The First Lessons in Intellectual Arithmetic was published at Boston in 1821. The book contained XX+195 pages of 15 sections instead of chapters and was divided into two parts. The first part was made up of the text proper, while the second part consisted of the key, or explanation of the plates, the tables of weights and measures, and the miscellaneous problems. A second edition followed in 1822. Colburn stated in the preface to the second edition that he believed better results would be obtained if all the sections were preceded by simple illustrative problems; hence the revision. In the first edition only those sections involving the fundamental operations were preceded by such problems. The two editions were similar in other respects: they both discussed numbers, the four fundamental operations with numbers, common fractions, and had practical problems connected with the four fundamental operations and fractions.

The Sequel to the First Lessons, published in 1824, was Colburn's second publication. It was written to follow the First Lessons in Intellectual Arithmetic and was prepared for children of 8 to 10 years of age. The Sequel followed the same general plan as the First Lessons. The discussion of decimal fractions is the most striking feature of the book.

The third and last publication of Colburn was his Algebra with the title An Introduction to Algebra upon the Inductive Method of Instruction, which was published about 1826. This book consisted of 276 pages divided into 54 sections instead of chapters. It discussed the usual topics included in the present day elementary and advanced algebras for high school students. One striking difference was the fact that he did not discuss ratio and proportion. The purpose of this text was to make the transition from arithmetic to algebra as simple as possible. This like all the others was written on the inductive method. There are few rules given at first, but on the other hand were many simple practical problems involving the first fundamental ideas of algebra. These problems are stated so that the student will devise his own rules for working out the more general problems of addition, subtraction, multiplication, and division. In the preface the author states: "In fact, explanations rather embarrass than aid the learner, because he is apt to trust too much to them, and to neglect his own powers, and because the explanation is frequently not made in the way that would naturally suggest itself to him, if he were left to examine the subject by himself. The best mode, therefore, seems to be, to give examples so simple as to require little or no explanation, and let the learner reason for himself, taking care to make them more difficult as he proceeds. This method besides giving the learner confidence, by making him rely on his own powers, is much more interesting to him, because he seems to himself to be constantly making new discoveries. Indeed, an apt learner will frequently make original explanations much more simple than would have been given by the author.

This mode has also the advantage of exercising the learner in reasoning, instead of making him a listener, while the author reasons before him."

Colburn states that the method given for finding the coefficients in the binomial expansion is original, likewise the manner of treating and demonstrating the principle of summing series by differences.[5] The present writer has examined the literature rather carefully and has been unable to find these methods given prior to Colburn. The method for finding the coefficients in the binomial expansion is as follows: "The coefficient of the first term of every power is 1. The coefficient of the second term of every power is formed by adding together the coefficients of the first and second terms of the preceding power. The coefficient of the third term of every power is formed by adding

[5]See *The Boston Journal of Philosophy and the Arts,* No. 5, May, 1825.

together the coefficients of the second and third terms of the preceding power. . . . . . And so on of the rest."

The method of summation of series is as follows: "Let 1, a, b, c, d, . . . , l be a series of any order, such that the sum of n terms may be found by multiplying the (n+1)th term by n, and dividing the product by m. If l is the (n+1)th term, and s the sum of all the terms, we will have by hypothesis

$$s=\frac{nl}{m}, \text{ and } m\,s=nl. \quad ''$$

In the discussion on logarithms we do not find the terms mantissa or cologarithm used. Instead of cologarithm Colburn uses the arithmetic complement, the rule for which is stated as follows: "The arithmetic complement is found by subtracting the logarithm from the logarithm of 1, which is zero, but which may always be represented by $-1+1$, $-2+2$, etc. It must always be represented by such a number that the logarithm of the number may be subtracted from the positive part. That is, it must always be equal to the characteristic of the logarithm to be subtracted, plus 1; for 1 must always be borrowed from it, from which to subtract the fractional part."

In the discussion of involution the radical sign is explained but not used. Fractional exponents are used instead of the radical sign.

One can barely conceive that such a far-reaching change in text-book writing should occur so suddenly as that made by Colburn. In the United States there was not a gradual change from the old deductive texts to the inductive texts. To the contrary, on the appearance of the text of Colburn many others of a similar nature followed, and consequently there came about a change in the methods of teaching both arithmetic and elementary algebra. It should be borne in mind that the changes in the methods of teaching these subjects did not take place as rapidly as the change in the writing of the text-books.

A casual perusal of the First Lessons in Intellectual Arithmetic and the Sequel to the First Lessons will readily show that the author had a firm grasp of the workings of the human mind, in particular that of the small child. His texts were worked out on a purely inductive rather than a deductive method. At the very beginning he introduces the pupil to a topic by means of simple practical problems. He always states the concrete before the abstract and does not give any symbolism until the four fundamental operations have been

mastered. His texts were the first published in this country to make use of mental arithmetic and to give sets of problems for oral drill.

The general scheme of Colburn's texts was as follows: to present the pupil with a difficulty—in many cases this may be a very simple difficulty, but the student at least has a task; he feels a need; he is then to form his own rules and draw his own conclusions. In all of his texts Colburn emphasizes the necessity of instructing the child how to study. He also suggests the monitorial system of instruction; that is, of letting the brighter, more rapid students work with and assist the slower students in their work in arithmetic.

A final important characteristic of Colburn was his ability to write texts that were interesting to small children. His problems were types that had been drawn from the child's everyday experience, and the style of his writing was simple and easily understood.

EUCLID AND THE AXIOM

As has been pointed out before, Euclid avoided the word axiom, and I believe with Grassman, that its omission in the ELEMENTS is not accidental, but the result of well-considered intention. The introduction of the term among Euclid's successors is due to the nature of geometry and the conditions through which its fundamental notions originate.

It may be a flaw in the Euclidean ELEMENTS that the construction of the plane is presupposed, but it does not invalidate the details of his glorious work which will forever remain classical.—Paul Carus in "*The Foundations of Mathematics.*"

WHAT JUSTIFIES MATHEMATICS?

If it is safe to trace back to any single man the origin of those conceptions with which pure mathematical analysis has been chiefly occupied during the nineteenth century and up to the present time, we must, I think, trace it back to Jean Baptiste Joseph Fourier (1768-1830). Fourier was first and foremost a physicist, and he expressed very definitely his view that mathematics only justifies itself by the help it gives towards the solution of physical problems, and yet the light that was thrown on the general conception of a function and its "continuity", of the "convergence" of infinite series, and of an integral, first began to shine as a result of Fourier's original and bold treatment of the problems of the conduction of heat. This it was that gave the impetus to the formation and development of the theories of functions.—G. Cantor in *The Theory of Transfinite Numbers*.

The Solution of Algebraic Equations by Infinite Series[1]

By ARTHUR J. LEWIS
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The purpose of this article is to outline methods of expressing all the roots of an algebraic equation by infinite series, and to formulate conditions of convergence for these series.

While the difficulty of solving algebraic equations, of degree less than five, by radicals depends chiefly upon the degree of the equation, that of solving equations by series is dependent, rather, upon the number of terms in the equation. In this paper, we shall consider trinomial, quadrinomial, and multinomial equations.

I. TRINOMIAL EQUATIONS

While J. H. Lambert[2] was probably the first to find a root of a trinomial equation by means of an infinite series, Lagrange[3] was undoubtedly the first to write a general series which would give all the roots of such an equation.

In his study of equations, Lagrange developed the following theorem[4] which has since borne his name:

If r is a root of the equation

$$x = a + m(x) \tag{1}$$

then any algebraic function of r, as f(r), may be expressed by the series

$$f(r) = f(a) + \sum_{s=1}^{\infty} \frac{1}{s!} \frac{d^{s-1}}{da^{s-1}} \left\{ \frac{s}{m(a)} f'(a) \right\} \tag{2}$$

The following development illustrates the use of Lagrange's theorem in solving trinomial equations and is, essentially, the same as that of P. A. Lambert[5] and others. The final result, however,

[1]Abstract of Chapter II of the Author's Thesis for Ph.D., University of Colorado.
[2]Acta Helvetica (1758), pp. 128-168.
[3]Lagrange, Oeuvres, Vol. III, pp. 5-73.
[4]For proof of Lagrange's theorem, see Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 404.
[5]Proceedings of the American Philosophical Society Vol. 47 (1908) pp. 111-135. Bulletin American Mathematical Society, Vol. 14 (1908) pp. 167-177.

appears to the author to be a more convenient form than those previously given.

Consider the general trinomial equation

$$az^n - bz^k - c = a \prod_{h=1}^{n} (z - z_h) = 0,$$

where a, b, and c are complex numbers, n and k are positive integers $n > k$. If we let $z = x^{1/n}$, equation (3) becomes

$$x = \frac{c}{a} + \frac{b}{a} x^{k/n}$$

This is in the form of equation (1), hence, we can substitute in Lagrange's formula (2). To simplify the work of substitution, we note the following:

a. Assuming $c/a = re^{i\theta}$, the roots of the binomial equation

$az^n - c = 0$ will be given by $a_h = r^{1/n} e^{(2h\pi + \theta)i/n}$ $(h = 1, 2, \ldots, n)$.

b. For $f(x) = x^{1/n}$, we have $f'(x) = \dfrac{1}{n} x^{1-n/n}$ and

$$f\left(\frac{c}{a}\right) = \left(\frac{c}{a}\right)^{1/n} = r^{1/n} e^{(2h\pi + \theta)i/n} = a_h \qquad (h = 1, 2, \ldots, n).$$

c. Comparing equations (1) and (4), we note that

$$m(x) = \frac{b}{a} x^{k/n}$$

hence $$\overline{m(x)}^{s} = \frac{b^s}{a^s} x^{sk/n}$$

and $$\frac{d^{s-1}}{dx^{s-1}} \left\{ \overline{m(x)}^{s} f'(x) \right\}$$

$$= \left(\frac{b}{an}\right)^{s} \left\{ (1 + sk - n)(1 + sk - 2n) \ldots (1 + sk - \overline{s-1}n) \right\} x^{1 + sk - sn/n}$$

Substituting in (2) we have

$$z_h = x^{1/n} = a_h + \sum_{s=1}^{\infty} \frac{1}{s!} \left(\frac{b}{an} \right)^s$$

$$\left\{ (1+sk-n)\,(1+sk-2n) \ldots (1+sk-\overline{s-1}\,n) \right\} a_h^{\,1+sk-sn}$$

The value of z_h may be put in a more convenient form by introducing the symbol

$$\left(\begin{matrix} 1+sk-n \\ s-1 \end{matrix} \right) = (1+sk-n)(1+sk-2n) \ldots (1+sk-\overline{s-1}\,n),\ s \geqq 2;$$

$$\left(\begin{matrix} 1+sk-n \\ s-1 \end{matrix} \right) = 1,\ s = 0,\ 1.$$

Finally, remembering $\left(\dfrac{c}{a} \right)^{1/n} = a_h$, we have

$$z_h = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{b}{cn} \right)^s \left(\begin{matrix} 1+sk-n \\ s-1 \end{matrix} \right) a_h^{\,1+sk} \qquad (h = 1, 2, \ldots, n) \tag{5}$$

It can be shown that a sufficient condition for this series to converge is that

$$\left| \frac{b^n}{a^k c^{n-k}} \right| < P^n, \text{ where } P^n = \frac{n^n}{k^k (n-k)^{n-k}} \tag{6}$$

When (5) converges, the n roots of (3) may be obtained by substituting in succession the n values of a_h.

In case this series does not converge, the roots of the given equation must be obtained by the use of two series, one giving k roots and the other $n-k$ roots. To obtain the k roots we write the equation in the form $bz^k = -c + az^n$. Assuming $-c/b = r_2 e^{i\theta_2}$ the roots of the equation $bz^k + c = 0$ are given by $b_h = r_2^{1/k} e^{(2h\pi + \theta_2)i/k} (h = 1, \ldots, k)$. Hence, replacing a, b, c, n, k, and a_h in (5) by b, a, $-c$, k, n, and b_h respectively, we obtain

$$z_h = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{-a}{ck} \right)^s \left(\begin{matrix} 1+sn-k \\ s-1 \end{matrix} \right) b_h^{\,1+sn} \qquad (h = 1, \ldots, k) \tag{7}$$

To obtain the remaining $n-k$ roots, we write the equation in the form $az^{n-k}=b+cz^{-k}$. Assuming $b/a=r_3e^{i\theta_3}$, the $n-k$ roots of $az^{n-k}=b$ are given by

$$c_h=r_3^{1/n-k}\,e^{(2h\pi+\theta_3)i/n-k}\quad (h=1,2,\ldots,n-k).$$

Hence, replacing a, b, c, n, k, and a_h in (5) by a, c, b, $n-k$, $-k$, and c_h respectively, we obtain

$$z_h=\sum_{s=0}^{\infty}\frac{1}{s!}\left\{\frac{c}{bn-bk}\right\}^s\left\{\begin{matrix}1-sk-(n-k)\\ s-1\end{matrix}\right\}c_h^{1-sk}\quad (h=1,\ldots,n-k)\quad (8)$$

If we apply the substitutions made to obtain series (7) and those to obtain series (8) to condition (6), we find that both of these series will converge if

$$\left|\frac{b^n}{a^kc^{n-k}}\right|>P^n \qquad (9)$$

In case

$$\left|\frac{b^n}{a^kc^{n-k}}\right|=P^n \qquad (10)$$

it is possible to show that the equation has multiple roots.

We are now prepared to outline the method of finding all the roots of any trinomial equation. Throw the equation in the form $az^n-bz^k-c=0$. Test by formulae (6), (9), and (10). If (6) holds, the n roots may be obtained by series (5). If (9) holds, k roots may be obtained by series (7), and the remaining $n-k$ roots by series (8). If (10) holds, there are multiple roots which should first be removed, and the resulting equation, if still a trinomial will come under one of the preceding cases. If it is not a trinomial equation, it will come under the more general case to be discussed later.

The application of the preceding formulae to the general quadratic equation $az^2-2bz-c=0$ is of interest. Let us assume first that condition (8) holds, that is $\left|\frac{b^2}{ac}\right|<1$.

Substituting in (5), we have

$$z_h=\sum_{s=0}^{\infty}\frac{1}{s!}\left(\frac{b}{c}\right)^s\binom{1+s-1}{s-1}a_h^{1+s}\qquad (h=1,2),$$

where a_h gives the two roots of the equation $az^2-c=0$, namely; $a_1=(c/a)^{1/2}$ and $a_2=-(c/a)^{1/2}$. Expanding, we get

$$z_1=\frac{c^{1/2}}{a^{1/2}}+\frac{b}{a}+\frac{b^2}{2a^{3/2}c^{1/2}}-\frac{b^4}{8a^{5/2}c^{3/2}}+\ldots+\frac{(2s-1)(2s-3)\ldots(3-2s)b^{2s}}{(2s)!a^{2s+1/2}c^{2s-1/2}}+\ldots$$

$$z_2=-\frac{c^{1/2}}{a^{1/2}}+\frac{b}{a}-\frac{b^2}{2a^{3/2}c^{1/2}}+\frac{b^4}{8a^{5/2}c^{3/2}}-\ldots-\frac{(2s-1)(2s-3)\ldots(3-2s)b^{2s}}{(2s)!\,a^{2s+1/2}c^{2s-1/2}}-\ldots$$

Solving the same equation by the quadratic formula, we have

$$z=\frac{b}{a}\pm\frac{1}{a}(ac+b^2)^{1/2}.$$

Since $|b^2|<|ac|$, this may be expanded by the binomial theorem into

$$z=\frac{b}{a}\pm\frac{1}{a}\left(a^{1/2}c^{1/2}+\frac{b^2}{2a^{1/2}c^{1/2}}+\ldots+\frac{(-1)(-3)\ldots(3-2s)}{2^s s!a^{2s-1/2}c^{2s-1/2}}+\ldots\right)$$

It is readily seen that this series agrees with the preceding, when we note that

$$\frac{(2s-1)(2s-3)\ldots(3-2s)}{(2s)!}=\frac{(2s-1)\ldots(1)(-1)(-3)\ldots(3-2s)}{2^s s!(1.3\ldots(2s-1))}$$

$$=\frac{(-1)(-3)\ldots(3-2s)}{2^s s!}.$$

Assuming that condition (9) holds, we have $\left|\frac{b^2}{ac}\right|>1$, and hence substituting in (6) and (7) in turn we have

$$z_1=\sum_{s=0}^{\infty}\frac{1}{s!}\left(\frac{a}{-c}\right)^s\binom{1+2s-1}{s-1}\left(\frac{-c}{2b}\right)^{1+2s}$$

$$= -\frac{c}{2b} + \frac{ac^2}{(2b)^3} - \ldots + \frac{(-1)^s(2s-2)!a^{s-1}c^s}{(s-1)!\,s!\,(2b)^{2s-1}} + \ldots;$$

$$z_2 = \sum_{s=0}^{\infty} \frac{1}{s!}\left(\frac{c}{2b}\right)^s \binom{1-s-1}{s-1}\left(\frac{2b}{a}\right)^{1-s}$$

$$= \frac{2b}{a} + \frac{c}{2b} - \frac{ac^2}{(2b)^3} + \ldots + \frac{(-1)^{s-1}s(s+1)\ldots(2s-2)c^s a^{s-1}}{s!\,(2b)^{2s-1}}$$

The reader may check these results by writing the roots of the given quadratic equation in the form $z = b/a \pm 1/a(b^2+ac)^{1/2}$ and applying the binomial theorem.

For an example with numerical coefficients, let us consider the equation

$$g(z) = z^5 + z^2 - 8 = 0. \tag{11}$$

Here $n=5$, $k=2$, $a=1$, $b=-1$, $c=8$.

Since formula (8) holds, we may obtain the five roots in a single series by use of formula (5)

Substituting in (5), we have

$$z_h = a_h - \frac{1}{40}a_h{}^3 + \frac{0}{2!(40)^2}a_h{}^5 + \frac{2\cdot 3}{3!(40)^3},a_h{}^7 + \ldots \quad (h=1,2,\ldots,5), \tag{12}$$

where $a_h = |8^{1/5}|(\cos 2h\pi/5 + i \sin 2h\pi/5$, $(h=1,2\ldots,5)$

$a_1 = 1.5157166(\cos 2\pi/5 + i \sin 2\pi/5) = 0.4683822 + 1.4415321\, i$

$a_2 = 1.5157166(\cos 2\pi/5 + i \sin 2\pi/5) = 0.4683822 + 1.4415321\, i$

$a_3 = 1.5157166(\cos 6\pi/5 + i \sin 6\pi/5) = -1.2262405 - 0.8909159\, i$

$a_4 = 1.5157166(\cos 8\pi/5 + i \sin 8\pi/5) = 0.4683822 - 1.4415321\, i$

$a_5 = 1.5157166(\cos 2\pi + i \sin 2\pi) = +1.5157166.$

Remembering that

$a_h{}^n = |8^{n/5}|(\cos 2nh\pi/5 + i \sin 2nh\pi/5)$, we get

$a_1{}^3 = -2.8171609 - 2.0467874\, i,$

$a_1{}^7 = -14.8690645 + 10.8030085\, i,$

$a_1{}^9 = 13.0480127 - 40.1576522\, i$

$a_1^{11} = 29.9765831 + 92.2584332\ i$

$a_2^{3} = 1.0760596 + 3.3117708\ i,$

$a_2^{7} = 5.6794772 - 17.4796329\ i,$

$a_2^{9} = -34.1601401 - 24.8187963\ i,$

$a_2^{11} = -78.4797126 + 57.0188530\ i,$ etc.

Substituting in equation (12), we get

$$z_1 = 0.5385834 + 1.4928530\ i$$
$$= 1.5870356(\cos 70°.1618 + i \sin 70°.1618),$$

$$z_2 = -1.2530650 + 0.8078378\ i$$
$$= 1.4908969(\cos 147°.1906 + \sin 147°.1906),$$

$$z_3 = -1.2530650 - 0.8078378\ i$$
$$= 1.4908969(\cos 147°.1906 - \sin 147°.1906),$$

$$z_4 = 0.53858334 - 1.4928530\ i$$
$$= 1.5870356(\cos 70°.1618 - i \sin 70°.1618),$$

$$z_5 = 1.4289638.$$

Substituting to check these results, we get

$g(z_1) = -0.000020 - 0.000031\ i,$

$g(z_2) = -0.000004 + 0.000020\ i$

$g(z_3) = -0.000020 + 0.000031\ i,$

$g(z_4) = -0.000004 - 0.000020\ i,$

$g(z_5) = -0.000006.$

Since the maximum error in $g(z_i)$ is less in absolute value than 37 units in the sixth decimal place, we find by differentials that the error in any root will be less than 2 units in the sixth decimal place.

II. QUADRINOMIAL EQUATIONS

The solution of quadrinomial equations by series has been treated by different authors, including Lagrange,[6] McClintock[7] and P. A. Lambert.[5] Of these Lambert alone attempts to give any general convergence conditions, and his conditions (28) and (32), of the article cited, fail because neither his X_q nor Z_q has an upper limit.

[7]Lagrange, Oeuvres, Vol. III, p. 73.
[8]*American Journal of Mathematics*, Vol. 17 (1895) pp. 85-110.

The following appears to be a satisfactory solution of the general quadrinomial equation

$$az^n - bz^k - cz^q - d = 0 \tag{21}$$

We assume a, b, c, d complex numbers, n, k, g positive integers with $n > k > g$.

For $w = z^n$, we have

$$w = d/a + m(w), \text{ where } m(w) = b/a\ w^{k/n} + c/a\ w^{q/n}.$$

Before applying formula (2) to this, we note the following:

a. If we let $d/a = r_i\ e^{i\Theta_1}$ the roots of $az^n - d = 0$ are given by $\alpha_h = r_1{}^{1/n} e^{(2h\pi + \Theta_1)i/n}$ $(h = 1, 2, \ldots, n)$.

b. For $f(w) = w^{1/n}$ we have $f(w) = 1/n\ w^{1-n/n}$, and

$$f\left(\frac{d}{a}\right) = \left(\frac{d}{a}\right)^{1/n} = \alpha_h \quad (h = 1, 2, \ldots, n),.$$

c. $$\overline{m(w)}^{\,r} = \sum_{s=0}^{r} \binom{r}{s} \frac{b^{r-s}c^s}{a^r}\ w^{kr-ks+gs/n}$$

$$\frac{d^{r-1}}{dw^{r-1}}\left(\overline{m(w)}^{\,r}\ w^{1-n/n} \right)$$

$$= \sum_{s=0}^{n} \binom{r}{s} \frac{b^{r-s}c^s}{a^r n^{r-1}} \binom{1+kr-ks+gs-n}{r-1} w^{1+kr-ks+gs-rn/n}.$$

Therefore, substituting in (2), we have the double series

$$z_h = \sum_{r=0}^{\infty} \frac{1}{r!(dn)^r} \sum_{s=0}^{r} \binom{r}{s} b^{r-s}c^s \binom{1+kr-ks+gs-n}{r-1} \alpha_h{}^{1+kr-ks+gs} \quad (h = 1, \ldots, n). \tag{22}$$

Replacing $r - s$ by u and s by v, this formula may be put in the more convenient form

$$z_h = \sum_{u,v=0}^{\infty} \frac{b^u c^v}{u!v!(dn)^{u+v}} \binom{1+uk+vg-n}{u+v-1} \alpha_h{}^{1+uk+vg} \quad (h = 1, \ldots, n) \tag{23}$$

To find sufficient conditions for convergence of this series we let R stand for the general term of series (22). Thus we obtain

$$R = \frac{1}{r!d^r n^r}\sum_{s=0}^{r}\left(\begin{matrix} r \\ s \end{matrix}\right) b^{r-s}c^s \left(\begin{matrix} 1+kr-ks+gs-n \\ r-1 \end{matrix}\right)\alpha_h^{1+kr-ks+gs}$$

Then $|R| < |K$, where

$$K = \frac{1}{r!d^r n^r}\left(\begin{matrix} 1+rk-n \\ r-1 \end{matrix}\right)\sum_{s=0}^{r}\left(\begin{matrix} r \\ s \end{matrix}\right) b^{r-s}c^s\alpha_h^{1+kr-ks+gs}$$

$$= \frac{\alpha_h^{(1+rk-n/n)}\,!\,(b\alpha_h^k + c\alpha_h^g)^r}{r!d^r n^{(1+rk-rn/n)}\,!}.$$

Applying Stirling's formula[9] and simplifying, we have

$$K = \frac{\alpha_h\left(\dfrac{k}{n}+\dfrac{1-n}{rn}\right)^{2+2rk-n/2n} e^{1+\epsilon_1-\epsilon_2-\epsilon_3}(b\alpha_h^k + c\alpha_h^g)^r}{nd^r r^{3/2r}\sqrt{2\pi}\left(\dfrac{k-n}{n}+\dfrac{1}{rn}\right)^{2+2rk-2rn+n/2n}}$$

where, as in the case of the trinomial equation, ϵ_1, ϵ_2, and ϵ_3 are the values ϵ assumes when n of this formula takes the values $1+rk-n/n$, r, and $1+rk-rn/n$ respectively.

Now applying Cauchy's radical test, we have

$$\lim_{r\to\infty}\sqrt[r]{K} = \frac{\left(\dfrac{k}{n}\right)^{k/n}(b\alpha_h^k + c\alpha_h^g)}{d\left(\dfrac{k-n}{n}\right)^{k-n/n}}$$

Therefore, remembering that $|R| < |K|$, we see that series (23) will converge absolutely when

$$\left|\frac{k^{k/n}(b\alpha_h^k + c\alpha_h^g)(n-k)^{n-k/n}}{nd}\right| \leqq 1,$$

[9]Stirling's formula $n! = \dfrac{n^{n+1/2}\sqrt{2\pi}e^{\epsilon}}{e^n}$, where $\epsilon = \dfrac{1}{12n} - \dfrac{1}{360n^3} + \cdots$

that is when $\left|\dfrac{b\alpha_h^k+c\alpha_h^g}{d}\right|^n \leqq \dfrac{n^n}{k^k(n-k)^{n-k}}=P^n.$

Since $\left|\dfrac{b_h^k\alpha+c\alpha_h^g}{d}\right| \leqq \dfrac{|b\alpha_h^k|+|c\alpha_h^g|}{|d|},$

a sufficient condition for convergence is given by

$$\left\{\frac{|b\alpha_h^k|+|c\alpha_h^g|}{|d|}\right\}^n \leqq P^n \quad [10] \tag{24}$$

In case $c=0$, equation (21) reduces to a trinomial and the formula for z_h reduces to that given for the trinomial equation. At the same time, the condition for convergence reduces to that given for the trinomial equation.

Before proceeding further, we recall that, for the trinomial equation, when $|b^n|<|a^kd^{n-k}P^n|$ the n roots can be obtained by a single series, while, if $|b^n|>|a^kd^{n-k}P|^n$, it is necessary to find the roots by two series, one giving k roots and the other $n-k$ roots. When b is so large in absolute value that the roots must be obtained in two series, we call b a *dominant* coefficient, otherwise, we call b a *weak* coefficient. To extend this definition to a multinomial equation, we always call the coefficients of the terms of highest and lowest degrees dominant. The coefficient of any other term is said to be dominant when we can find the number of roots equivalent to the difference in the degree of this term and that of the next preceding term having a dominant coefficient, by a single series. The exact meaning of this definition will be made clear by what follows.

In discussing the general quadrinomial equation, there are four possible cases:

Case I: b and c both weak.

This is the case just considered where the n roots are obtained by a single series using formula (23). The condition for convergence is given by (24).

Case II: b dominant and c weak.

In this case, we write equation (21) in the form

$$az^{n-k}=b+(dz^{-k}+cz^{g-k}).$$

[10]The conditions of convergence for the trinomial equation may be more readily obtained by this method than by the older methods.

If we assume that $b/a = r_2 e^{i\Theta_2}$ the roots of $az^{n-k} = b$ will be given by $\beta_h = r_2^{1/n-k} e^{(2h\pi+\Theta_2)i/n-k}$ $(h = 1, \ldots, n-k)$.

Replacing a, b, c, d, n, k, g, and α_h of formula (23) by a, d, c, b, $n-k$, $-k$, $g-k$, and β_h respectively, we have

$$z_h = \sum_{u,v=0}^{\infty} \frac{d^u c^v}{u!\, v!(bn-bk)^{u+v}} \begin{Bmatrix} 1-uk+vg-vk-(n-k) \\ u+v-1 \end{Bmatrix} \beta_h^{1+vg-uk-vk} \quad h = (1, \ldots, n-k) \tag{25}$$

At the same time, the condition for convergence becomes

$$\left\{ \frac{|d\beta_h^{-k}| + |c\beta_h^{g-k}|}{|b|} \right\}^{n-k} \leqq 1/P^n \tag{26}$$

We now write equation (21) in the form

$$bz^k = -d + (az^n - cz^g).$$

If we assume $-\ d/b = r_3\, e^{i\Theta_3}$, the k roots of $bz^k = -d$ will be given by $\gamma_h = r_3^{1/k}\, e^{(2h\pi+\Theta_3)i/k}$ $(h = 1, 2, \ldots, k)$.

Making the necessary transformations in formulae (23) and (24), we get

$$z_h = \sum_{u,v=0}^{\infty} \frac{a^u(-c)^v}{u!\, v!(-dk)^{u+v}} \begin{Bmatrix} 1+un+vg-k \\ u+v-1 \end{Bmatrix} \gamma_h^{1+un+vg} \quad (h = 1, \ldots, k) \tag{27}$$

with the condition for convergence

$$\left\{ \frac{|a\gamma_h^n| + |c\gamma_h^g|}{|d|} \right\}^k \leqq 1/P^n.$$

Case III: b weak and c dominant.

Write equation (21) in the form $az^{n-g} = c + (dz^{-g} + bz^{k-g})$.

If we assume $c/a = r_4 e^{i\Theta_4}$, the roots of $az^{n-g} = c$ will be given by $\delta_h = r_4^{1/n-g}\, e^{(2h\pi+\Theta_4)i/n-g}$ $(h = 1, \ldots, n-g)$.

If $|-g| \geqq |k-g|$ we replace a, b, c, d, n, k, g, α_h in formulae (23) and (24) by a, d, b, c, $n-g$, $-g$, $k-g$, and δ_h respectively, obtaining

$$z_h = \sum_{u,v=0}^{\infty} \frac{d^u b^v}{u!v!(cn-cg)^{u+v}} \begin{Bmatrix} 1-ug+vk-vg-(n-g) \\ u+v-1 \end{Bmatrix} \delta_h^{1-ug+vk-vg} \quad (h = 1, \ldots, n-g) \tag{29}$$

the corresponding condition for convergence becoming

$$\left\{\frac{|d\delta_h^{-g}|+|b\delta_h^{k-g}|}{|c|}\right\}^{n-g} \leqq \frac{(n-g)^{n-g}g^g}{n^n}. \tag{30}$$

If $|k-g|>|-g|$, we replace a, b, c, d, n, k, g, α_h by a, b, d, c, $n-g$, $k-g$, $-g$, δ_h respectively. These substitutions give the same value for z_h as the preceding, but change the conditions for convergence to the relation

$$\left\{\frac{|d\delta_h^{-g}|+|b\delta_h^{k-g}|}{|c|}\right\}^{n-g} \leqq \frac{(n-g)^{n-g}}{(k-g)^{k-g}(n-k)^{n-k}} \tag{31}$$

The remaining g roots may be obtained by writing equation (21) in the form $cz^g=-d+(az^n-bz^k)$.

If we let $-d/c=r_5e^{i\theta_5}$, the g roots of $cz^g=-d$ will be given by $\epsilon_h=r_5^{1/g}e^{(2h\pi+\theta_5)i/g}(h=1,\ldots,g)$

Making the necessary substitutions in (23) and (24), we find

$$z_h=\sum_{u,v=0}^{\infty}\frac{a^u(-b)^v}{u!v!(-dg)^{u+v}}\left\{\begin{matrix}1+un+vk-g\\ u+v-1\end{matrix}\right\}\epsilon_h^{1+un+vk} \quad (h=1,\ldots,g) \tag{32}$$

where the condition for convergence is given by

$$\left\{\frac{|a\epsilon_h^n|+|b\epsilon_h^k|}{|d|}\right\}^{g} \leqq \frac{(n-g)^{n-g}g^g}{n^n} \tag{33}$$

Case IV: b dominant and c dominant. Write (21) in the form $az^{n-k}=b+(dz^{-k}+cz^{g-k})$.

The value of z_h is given by (25) and the condition of convergence by (26).

For the next group of roots, write equation (21) in the form $bz^{k-g}=-c+(az^{n-g}-dz^{-g})$. If we assume the roots of $bz^{k-g}=-c$ are given by $\zeta_h=r_6^{1/k-g}e^{(2h\pi+\theta_6)i/k-g}$ $(h=1,\ldots,k-g)$.

If $|n-g|>|-g|$, we replace a, b, c, d, n, k, g, and α_h of (23) and (24) by b, a, $-d$, $-c$, $k-g$, $n-g$, $-g$, and ζ_h respectively, obtaining

$$z_h=\sum_{u,v=0}^{\infty}\frac{a^u(-d)^v}{u!v!(gc-kc)^{u+v}}\left\{\begin{matrix}1+un-ug-vg-(k-g)\\ u+v-1\end{matrix}\right\}\zeta_h^{1+un-ug-vg} \quad (h=1,\ldots,n-g). \tag{34}$$

with the condition for convergence

$$\left\{\frac{|a\zeta_h^{n-g}|+|d\zeta_h^{-g}|}{|c|}\right\}^{k-g} \leqq \frac{(k-g)^{k-g}(n-k)^{n-k}}{(n-g)^{n-g}} \tag{35}$$

In case $|-g| \leqq |n-g|$, the condition for convergence will be

$$\left\{\frac{|a\zeta_h^{n-g}|+|d\zeta_h^{-g}|}{|c|}\right\}^{k-g} \leqq \frac{(k-g)^{k-g}g^g}{k^k} \tag{36}$$

To find the remaining g roots, we use formula (32) together with condition (33).

As an example, consider the equation

$f(z)=z^5-9z^3-z-3=0$

Here $a=1$, $b=9$, $c=1$, $d=3$, $n=5$, $k=3$, $g=1$. (37)

Since conditions (26) and (28) are satisfied, this comes under case II. Substituting in formula (25) we have

$$z_h=\sum_{u,v=0}^{\infty}\frac{3^u}{u!\,v!18^{u+v}}\binom{1-3u-2v-2}{u+v-1}\beta_h^{1-3u-2v} \qquad (h=1,\,2),$$

where $\beta_1=3$ and $\beta_2=-3$

$$z_h=\beta_h+\frac{1}{18}\beta_h^{-1}-\frac{5}{2!\,18^2}\beta_h^{-3}+\frac{7\cdot 9}{3!\,18^3}\beta_h^{-5}-\frac{9\cdot 11\cdot 13}{4!\,18^4}\beta_h^{-7}+\dots$$

$$+\frac{3}{18}\beta_h^{-2}-\frac{3\cdot 6}{18^2}\beta_h^{-4}+\frac{3\cdot 8\cdot 10}{2!18^3}\beta_h^{-6}-\frac{3\cdot 10\cdot 12\cdot 14}{3!\,18^4}\beta_h^{-8}$$

$$-\frac{3\cdot 7}{2!18^2}\beta_h^{-5}+\frac{3\cdot 9\cdot 11}{2!18^3}\beta_h^{-7}-\frac{3\cdot 11\cdot 13\cdot 15}{2!2!18^4}\beta_h^{-9}+\dots$$

$$+\dots$$

$z_1=3.035743$

$z_2=3.000000$

To get the remaining 3 roots, we substitute in formula (27).

$$z_h = \sum_{u,v=0}^{\infty} \frac{(-1)^u}{u!v!9^{u+v}} \begin{Bmatrix} 1+5u+v-3 \\ u+v-1 \end{Bmatrix} \gamma_h^{1+5u+v},$$

where

$$\gamma_h = \left(\frac{1}{3}\right)^{1/5} e^{(2h+1)\pi i/3} \qquad (h=3, 4, 5).$$

$$z_h = \gamma_h + \frac{1}{9}\gamma_h^2 - \frac{2}{3!9^3}\gamma_h^4 + \frac{2\cdot 4}{4!9^4}\gamma_h^5 - \frac{4\cdot 2\cdot 5\cdot 8}{6!9^7}\gamma_h^7 + \dots$$

$$-\frac{1}{9^2}\gamma_h^6 - \frac{4}{9^3}\gamma_h^7 - \frac{5\cdot 2}{2!9^4}\gamma_h^8 + \frac{7\cdot 4\cdot 2}{4!9^5}\gamma_h^{10} - \frac{8\cdot 5\cdot 2\cdot 4}{5!9^6}\gamma_h^{11} + \dots$$

$$+\frac{8}{2!9^2}\gamma_h^{11} + \frac{9\cdot 6}{2!9^3}\gamma_h^{12} + \frac{10\cdot 7\cdot 4}{2!2!9^4}\gamma_h^{13} + \frac{11\cdot 8\cdot 5\cdot 2}{2!3!9^5}\gamma_h^{14} + \dots$$

$$-\frac{13\cdot 10}{3!9^3}\gamma_h^{16} - \frac{14\cdot 11\cdot 8}{3!9^4}\gamma_h^{17} - \frac{15\cdot 12\cdot 9\cdot 6}{3!2!9^5}\gamma_h^{18} - \dots$$

$$+ \dots$$

Finding successive powers of γ_h and substituting in (38) we find

$z_3 = 0.306883 + 0.642639\, i$,

$z_4 = -0.649515$,

$z_5 = 0.306883 - 0.642639\, i$.

To check these results, we substitute in the original equation obtaining

$f(z_1) = 0.000168$,

$f(z_2) = 0.000000$,

$f(z_3) = 0.000002 + 0.000026\, i$,

$f(z_4) = 0.000017$,

$f(z_5) = 0.000002 - 0.000026\, i$.

The test by differentials shows that the error in any root does not exceed two units in the sixth decimal place.

III. GENERAL MULTINOMIAL EQUATIONS

Let us now consider the solution of the general equation

$$a_n z^n - a_k z^k - a_g z^g - \ldots - a_b z^b a_o = 0, \tag{39}$$

where the coefficients are complex numbers and the exponents are positive integers.

We may write (39) in the form

$$z^n = a_o/a_n + 1/a_n(a_k z^k + a_g z^g + \ldots + a_b z^b). \tag{40}$$

Substituting w for z^n in (40) we obtain

$$w = a_o/a_n + m(w)$$

where $m(w) = 1/a_n(a_k w^{k/n} + a_g w^{g/n} + \ldots + a_b w^{b/n})$.

This is in the required form for the application of Lagrange's formula. We first note that for $a_o/a_n = re^{i\Theta}$ the n roots of $a_n z^n - a_o = 0$ are given by $\alpha_h = r^{1/n} e^{(2h\pi + \Theta)i/n}$ $(h = 1, \ldots, n)$ and that

$$\overline{m(w)}^{\,r} = \frac{1}{a_n^r} \sum_{p,q,\ldots,v=0}^{r} \frac{r!a_k^p a_g^q \ldots a_b^v}{p!\,q! \ldots v!} (p+q+\ldots+v=r)$$

If we let $f(w) = w^{1/n}$ and substitute in (2) we obtain

$$z_h = \sum_{r=0}^{\infty} \sum_{p,q,\ldots,v=0}^{r} \frac{a_k^p\, a_g^q \ldots a_b^v}{p!q!\ldots v!n^r a_n^r} \begin{pmatrix} 1+pk+\ldots+vb-n \\ r-1 \end{pmatrix} \left(\frac{a_o}{a_n} \right)^{1+pk+\ldots+vb-rn/n} (p+\ldots+v=r)$$

or

$$z_h = \sum_{p,q,\ldots v=0}^{\infty} \frac{a_k^p a_g^q \ldots a_b^v}{p!q!\ldots v!(a_o n)^{p+q+\ldots+v}} \begin{pmatrix} 1+pk+\ldots+vb-n \\ p+q+\ldots+v-1 \end{pmatrix} \alpha_h^{1+pk+\ldots+vb} (h=1,\ldots,n \tag{41}$$

To find the conditions for convergence, we let

$$R = \sum_{p,\ldots v=0}^{r} \frac{a_k^p \ldots a_b^v}{p!\ldots v!(a_o n)^r} \begin{pmatrix} 1+pk+\ldots+vb-n \\ r-1 \end{pmatrix} \alpha_h^{1+pk+\ldots+vb} \quad (p+\ldots+v=r).$$

Then $|R| < |K|$ where

$$K = \frac{\alpha_h}{r!(a_o n)^r} \begin{pmatrix} 1+rk-n \\ r-1 \end{pmatrix} (a_k \alpha_h^k + \ldots + a_b \alpha_h^b)^r.$$

Applying Stirling's formula to this and applying Cauchy's radical test to the result, as in the preceding cases, we get

$$\lim_{r\to\infty} \sqrt[r]{K} = \frac{\left(\dfrac{k}{n}\right)^{k/n}(a_k\alpha_h{}^k + a_g\alpha_h{}^g + \ldots + a_b\alpha_h{}^b)}{a_o\left(\dfrac{k-n}{n}\right)^{k-n/n}}$$

Since $\lim_{r\to\infty} \sqrt[r]{R} < \lim_{r\to\infty} \sqrt[r]{K}$, the series converges when

$$\left|\frac{a_k\alpha_h{}^k + \ldots + a_b\alpha_h{}^b}{a_o}\right|^n \leqq \frac{n^n}{k^k(n-k)^{n-k}}$$

Finally a sufficient condition for absolute convergence is given by the relation

$$\left\{\frac{|a_k\alpha_h{}^k| + \ldots + |a_b\alpha_h{}^b|}{|a_o|}\right\}^n \leqq \frac{n^n}{k^k(n-k)^{n-k}} \tag{42}$$

In case condition (42) is not satisfied, we take up the different cases which arise when we put equation (39) in the forms,

$$\begin{aligned} z^{n-k} &= a_k/a_n + w_{nk}(z), \\ z^{n-g} &= a_g/a_n + w_{ng}(z), \\ &\ldots\ldots\ldots\ldots \\ z^{k-g} &= -a_g/a_k + w_{kg}(z), \\ &\ldots\ldots\ldots\ldots \\ z^b &= -a_o/a_b + w_b(z). \end{aligned} \tag{43}$$

Corresponding to each of these equations, we can set up a series similar to (41), which, if convergent, will give as many of the roots of (39) as the degree of z in the left-hand member of the particular equation of (43) which we are using. Before writing these series, however, we apply tests similar to (42) in order to determine which will converge. In writing equation (43) w(z) will frequently contain negative powers of z. In making the tests for convergence, it is to be understood that the absolute value of the exponent of largest numerical value in w(z) is to correspond to the k of (42).

We have thus outlined methods of solving any equation by infinite series provided the conditions of convergence, here given, are satisfied.

Roots of Matric Equations

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By the application of a simple and well-known principle, the theorem that complex roots of algebraic equations with real coefficients occur in pairs, can be easily extended to matric equations. The principle and its application are elementary and can serve as illustrative material in a course in algebra in which the subject of matrices is included.

If a and b are complex numbers and $\bar{a}$ and $\bar{b}$ are their respective conjugates then we know that

$$\overline{a.b} = \bar{a}.\bar{b} \qquad \text{and} \qquad \overline{a \pm b} = \bar{a} \pm \bar{b}$$

This property of complex numbers is possessed, however, by some classes of matrices. For instance, if A and B are symmetric matrices and $\bar{A}$ and $\bar{B}$ their respective conjugates; or if A and B are Hermitian matrices and $\bar{A}$ and $\bar{B}$ their respective conjugates, it can be easily verified in either case that

$$(1) \qquad \overline{A.B} = \bar{A}.\bar{B} \qquad \text{and} \qquad \overline{A \pm B} = \bar{A} \pm \bar{B}.$$

Suppose that any number of matrices of the same order m and subject to (1) undergo a finite number of additions and multiplications producing as a result the matrix M. We can express this in the form

$$(2) \qquad P(A,B,\ldots K) \equiv M$$

in which the left member represents a polynomial formed by combining A, B, ..., K additively and multiplicatively. It readily follows from (1) that*

$$(3) \qquad P(\bar{A},\bar{B},\ \ldots,\ \bar{K}) = \bar{M}.$$

Let the square matrix A of order m satisfy the equation

$$F(X) = C_0X^n + C_1X^{n-1} + \ \ldots\ + C_n = 0$$

*For the null matrix, $O, \bar{O} = 0$.

where the C_i are square matrices of order m. Moreover let A and the C_i satisfy (1). Thus $F(A) \equiv 0$ and by virtue of the remark made above $\overline{F}(\overline{A}) \equiv 0$ where $\overline{F}(X)$ is obtained from $F(X)$ by replacing the C_i by $\overline{C}_i$. Consequently the fact that (3) follows from (2) implies the following:

If the square matrix A satisfies an equation $F(X)=0$ having matric coefficients of the same order as A, and if these coefficients together with A satisfy conditions (1), then $\overline{A}$ satisfies the equation $\overline{F}(X)=0$.

If we write

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i$$

it can be shown that any complex number $a+ib$ can be represented by the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

while its conjugate $a-ib$ is given by the transpose of the preceding matrix, namely

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Evidently these matrices, though neither Hermitian nor symmetric, satisfy conditions (1). We can therefore draw the corollary:

If A is a complex number, or an Hermitian matrix of order m or a symmetric matrix of order m, and if A satisfies an algebraic equation $F(X)=0$, the coefficients of which are respectively complex numbers, Hermitian matrices of order m or symmetric matrices of order m, then $\overline{A}$ will satisfy the equation $\overline{F}(X)=0$, obtained from $F(X)=0$ by replacing the coefficients by their conjugates.

Furthermore if we write

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = i$$

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = j \qquad \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = k$$

we can represent the quaternion $q = a + ib + jc + kd$ in the form of a matrix*

$$\begin{bmatrix} a & b & -c & -d \\ -b & a & d & -c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}$$

The transpose of this matrix is the conjugate of q, namely, $a - ib - jc - kd$. Since the conjugate of the product of two quarternions equals the product of the conjugates in reverse order, these matrices satisfy the conditions

$$(1') \qquad \overline{A.B} = \overline{B}.\overline{A}, \qquad \overline{A \pm B} = \overline{A} \pm \overline{B}.$$

If for these matrices

$$P(A, B, \ldots, K) \equiv M$$

then

$$P'(\overline{A}, \overline{B}, \ldots, \overline{K}) \equiv \overline{M}$$

where P' is obtained from P by writing the factors of each term in P in the reverse order. This results in the following theorem:

If a quaternion q satisfies the equation

$$C_0X^n + C_1X^{n-1} + \ldots + C_n = 0$$

where the C_i are quaternions, the conjugate of q will satisfy the equation $X^n\overline{C}_0 + X^{n-1}\overline{C}_1 + \ldots + \overline{C}_n = 0$.

*Veblen and Young, Projective Geometry, vol. 2, (1918), p. 337.

SOUTHERN INTERCOLLEGIATE MATHEMATICS ASSOCIATION

My dear Dr. Maizlish:

I recently received, at your direction, a copy of the folder of the Southern Intercollegiate Mathematics Association, for which please accept my thanks and my hearty congratulations over the success of this undertaking on your part. It was a good idea and it evidently has proved itself well worthwhile. Yours very sincerely,

H. E. SLAUGHT.

A Graphic Solution of the Equation

$a \cos \Theta + b \sin \Theta = c$.

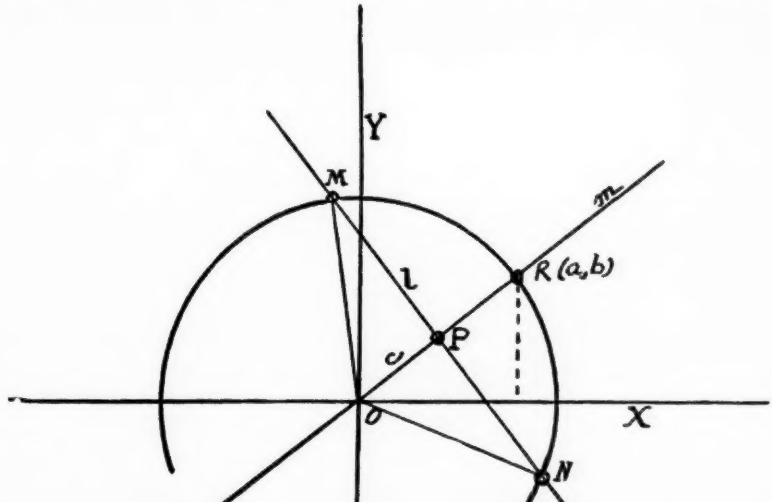
By H. T. R. AUDE
Colgate University

The following graphical method for the solution of equations of the form $a \cos \Theta + b \sin \Theta = c$ is easily performed with the ruler and compasses. It can readily be followed through and justified by a student who is toward the end of a course in trigonometry. For that reason the proof of this graphic method is not given. Some teachers may find it a suitable problem to assign to the better students. One of these may then show and prove it before the class.

For convenience write the equations so that a is positive. (This is really not necessary.) In the plane, using rectangular coordinates where OX is the positive direction of the X-axis, draw the line m through the two points O, the origin, and R, the point whose coordinates are (a, b). Take the direction on m from O to R as positive and lay off $OP = c$ along the line m. Draw the line l perpendicular to m at P. Next, draw the circle whose center is at O with radius equal to OR. Suppose the circle cuts the line l in the points M and N; then the angle Θ is equal to angle XOM or angle XON.

This construction holds whether the numbers a, b and c are positive, negative, or zero. It shows that for angles between 0° and 360° there are two distinct solutions, one, or no solution, according as c is numerically less than, equal to, or greater than $\sqrt{a^2+b^2}$.

This graphic solution may also be of value to an instructor who may thus locate the angles which will satisfy an equation of this type.



The Teacher's Department

Edited by
JOSEPH SEIDLIN

THE MATHEMATICS MEETING OF HAMILTON COLGATE, AND SYRACUSE

For several years the departments of mathematics of Hamilton, Syracuse, and Colgate have held two annual conferences. One is usually held in October, the other in May. The meeting place has been, in turn, one of these three institutions. There is no organization; no dues, laws or officers. The group was brought together by the friendly feelings among colleagues and has been held together by their common interest in mathematics. The aim has been to become better acquainted, to compare courses, to consider teaching problems and methods of instruction; just to discuss the common problems in the teaching of mathematics; merely to "talk shop" and have a few social hours together.

Each time there has been a program of which the department that is host has charge. But members from the other two departments take part in the program which has been arranged previously. A part of the time of a meeting is usually devoted to teaching problems, content of a course of study, changes in courses, coordination of subject matter, errors and vagaries of students, etc. Another part of the program has usually been given to the subject matter of mathematics where some of the members have presented topics from their own fields of study.

A more detailed account of the last meeting which was held this year on October 19th, at Colgate, may be of interest. The guest speaker, Professor Joseph Seidlin of Alfred University, was introduced by Dr. A. W. Smith, head of the department at Colgate. Mr. Seidlin addressed the group on class room procedure. In so doing he related how he came to write the book, "A Critical Study of the Teaching of Elementary College Mathematics," (Columbia University, 1931), and referred several times to parts of it. Those present were quite familiar with the book and appreciated listening to its author. Questions and answers crowded the allotted time. The recess following allowed the members to enjoy a tea served by the wives of the Colgate staff. At this time President Cutten of Colgate joined the group and welcomed the visiting members from Hamilton College, Syracuse

University, and Professor and Mrs. Seidlin. The meeting was then continued by a round table conference inspired by the following quotation from R. J. Jeffrey's article on Productive Scholarship in the Undergraduate College in the *American Mathematical Monthly*, June-July, 1935:

> ". . . . In every course we teach we find parts of the work which for some reason do not seem to go over. On these points class after class stalls for an unreasonably long time. There is a tendency for us to accept this situation as inevitable; to consider that there is something inherently difficult in the part of the work in question; or to let it go with the statement that the students we are getting nowadays are not up to what they should be. Of late, however, I have come to see this problem in a different way. Perhaps my own experience has led me to attribute undue emphasis to it. At any rate I have become convinced that when the situation we are describing arises the trouble is much more likely to be with our methods than with the work or the students. And once this fact is faced the problem does resolve itself into one of creative effort. We have to think the whole thing through from every possible angle, and keep working at it until we devise a method of presentation by which the work goes over easily and at once. To accomplish this one's ingenuity is sometimes pretty heavily taxed. But the longer I teach the more I am convinced that there is room for much of this sort of creative effort. . . ."

The group then adjourned to Colgate Inn for dinner. The discussions were continued in smaller groups at the table. An invitation for the spring meeting in 1936 was extended by Professor H. S. Brown in behalf of the department of mathematics at Hamilton College.

H. T. R. AUDE, Colgate College.

STUDENT TUTORIAL SYSTEM IN FRESHMAN MATHEMATICS IN ALBION COLLEGE

For several years the tutorial system in freshman mathematics has been used in Albion College for the purpose of helping those students who have difficulties in adjusting themselves to the new environment. Seniors and juniors who are majoring in this field are selected as tutors, the supervision of the work being in the hands of the departments of Mathematics and Education.

During the first two weeks of the semester all students who are having difficulties are advised to meet with the tutors for special help

and drill. During this period informational tests are given. The results of these tests are not recorded, but serve rather to give information, both to the student and to the instructor. Students showing a marked deficiency are then assigned to special tutors, not more than five or six to each. In this manner each student so assigned receives about four hours of special training each week. By setting the plan into operation very early in the semester, many a student is saved from the discouragement which so often comes after a few weeks of college mathematics. One of the major functions of the tutor is to help the student develop and establish proper study habits. During the first days of his college career a student hesitates to make his troubles known to the instructor, but frequently he will make a confidant of the junior or senior who is assisting him.

It has been found that voluntary work on the part of the tutor is not always satisfactory. For this reason an arrangement has been made whereby credit is given by the Department of Education. The tutor is requested to have frequent conferences with the instructor in education as well as the instructor in mathematics. The tutor is required to keep a systematic record concerning each student with whom he works. At the end of the semester a case study of each freshman is prepared, showing his difficulties as well as the methods used in helping him to overcome these difficulties.

To quote from some of these reports, we find the following statements: "Mr. W. is entirely too slow for the normal course of instruction. When given twice as much time as any other student he does well. When he has grasped a process completely he does not forget it. I question his choice of engineering as a career."

"This student is very responsive to assistance. He has a will of his own; he learns readily but not too well and therefore easily forgets what he learns. He is not too certain of remaining in engineering. I recommend, however, that he be encouraged to continue. He is rather young, but in time will develop the steadiness and seriousness which he seems to lack at present."

"In the case of this student I found that his difficulties date back to his preparatory mathematics. He claims that his high school teacher of mathematics took no interest in him, and showed none in mathematics. Another and a real difficulty arises from the fact that he does not have time to study. He is compelled to work many hours per day to earn enough to remain in college."

"This student presents a real problem. She was valedictorian of her class in high school. The reason for her failure seems to lie in the fact that she is having difficulty in adjusting herself to college life.

In tutoring her I find that she seems to grasp the points in question very quickly, and seems to be very sure of herself. I believe she knows the work but because of maladjustment shows up to least advantage on examination."

"This student seems to think he is doing me a favor by coming for help. He has no interest in college, he would not be here were he free to do as he pleases."

From these statements we note that the tutor often obtains reactions which are very beneficial as information not only to the department but also to the Dean of college and to the members of the faculty engaged in special personnel work. More recently the departments of English and Modern Language have adopted the plan. We recommend it to other departments of mathematics.

E. R. SLEIGHT, Albion, Mich.

THE RANGE OF GREEK MATHEMATICS

Of all the manifestations of the Greek genius none is more impressive and even awe-inspiring than that which is revealed by the history of Greek mathematics. Not only are the range and the sum of what the Greek mathematicians actually accomplished wonderful in themselves; it is necessary to bear in mind that this mass of original work was done in an almost incredibly short space of time, and in spite of the comparative inadequacy (as it would seem to us) of the only methods at their disposal, namely, those of pure geometry supplemented where necessary, by the ordinary arithmetical operations.—F. L. Heath in *Greek Mathematics.*

BEGINNINGS OF THE CALCULUS

In spite of the great abilities of Newton and Leibniz the underlying principles of the calculus as exposed by them seem to us from our modern viewpoint, as indeed to their contemporaries and immediate successors, somewhat vague and confusing. The difficulty lies in the lack of clearness at that early time, and for more than a century thereafter, in the conceptions of infinitesimals and limits upon which the calculus rests, a difficulty which has been overcome only by the systematic study of the theory of limits inaugurated by Cauchy (1989-1857) and continued by Weierstrass (1815-97), Riemann (1826-66), and many others—Gilbert Ames Bliss, *Calculus of Variations.*

Notes and News

Edited by

L. J. ADAMS and I. MAIZLISH

WORLD-WIDE MATHEMATICS

The American Mathematical Society will hold its next summer meeting on September 1-5, 1936, in connection with the Tercentenary Celebration of Harvard University. Professor E. W. Chittenden will deliver the Colloquium Lectures, on *Topics in General Analysis*. A joint meeting will be held with the American Astronomical Society. The Mathematical Association of America will meet on August 31.

Dr. R. C. Tolman, professor of mathematical physics at the California Institute of Technology, has been elected president of the Pacific Division of the American Association for the Advancement of Science.

National Research Fellowships in mathematics for the year 1935-36 have been awarded to the following: J. A. Clarkson, Norman Levinson, W. T. Martin, F. J. Murray, S. B. Myers, J. B. Rosser and G. C. Webber.

Econometrica for October, 1935 includes the annual survey of statistical technique by Paul Lorenz, with the title *Trends and Seasonal Variations*. There is a valuable bibliography divided into two parts:

1. Orthogonal rational integral functions for the analysis of statistical series.
2. Monographs on seasonal variations.

Students of the history of mathematics will find many useful bibliographical notes and abstracts in *Isis*. This is a journal edited by an American, written in English and published by The Saint Catherine Press, 51, Rue du Tram, Bruges, Belgium. Although nominally dealing with the history of science, there is much to be found which is purely mathematical.

The well known *Jahrbuch über die Fortschritte der Mathematik* contains a list of over three hundred journals from all over the world in *Band 56, Zweiter Halbband*. This list includes the names of the groups publishing the respective journals and the addresses of the publishers.

The International Mathematics Congress will meet in Oslo, Norway from July 13 to 18, 1936. In addition to the general meetings there will be departmentalized conferences, grouped as follows: Algebra and theory of numbers, analysis, geometry and topology, theory of probability and statistics, astronomy, mechanics and mathematical physics, and the philosophy, history and pedagogy of mathematics.

The Tôhoku Mathematical Journal, described in these pages last month, has included an index to its volumes 31-40 inclusive in the April, 1935 number. The index is arranged alphabetically by authors, giving the titles of articles below the writers' names. Research workers will find the index to this voluminous publication very useful.

Sphinx, revue mensuelle des questions récréatives, has found well-deserved favor with those interested in mathematical recreations. It is written in French, and is published at 75 Rue Philippe Baucq Bruxelles, Belgium. A section is devoted to cryptograms.

Those interested in mathematical models who have not seen the catalog of G. Cussons, Ltd., Manchester, England have a treat in store. The illustrations are profuse and the explanations are complete. The index is suggestive, with the following major topics: plane figures, polyhedra, surfaces of the second order, surfaces of revolution and screw surfaces, space curves, link systems, models for teaching calculation, models for teaching plane geometry and models for teaching solid geometry.

It is illuminating to compare the examination questions published occasionally in *L'Education Mathématique* with those of our own Regents, West Point and Annapolis. This magazine is a publication of Librairie Vuibert, Bouluvard Saint-Germain, 63, Paris, 5^e. The elementary problem department is also worthwhile. A very similar journal of the same publisher is *Revue de Mathématique Spéciales,* which deals with mathematics of the freshman and sophomore level. The editors of the latter are members of the faculties of various French lycées.

An old article on *Taxation as an Instrument for Modifying Inequalities in Distribution* by M. R. Doresamiengar in the Journal of the Indian Mathematical Society (190, Mount Road, Madras) for August, 1931 takes on new significance in view of the current effort to mathematize the ideas of economics. The choice for relation between welfare and income is $y = k\sqrt{x}$, on the assumption that marginal welfare does not decline so rapidly as marginal utility.

An often repeated suggestion of E. T. Bell of the California Institute of Technology is that someone publish lists of, for example three hundred frequently occurring words in each of several foreign languages. Such lists would be of great value to research workers in the mathematical field. Certainly French, German, Italian, Spanish and Russian lists would be welcomed. Mathematical terms unlike the corresponding English words should be included.

A manufacturing concern in Manchester, England has recently made a machine for the mechanical integration of differential equations. The theory of the instrument was described at a meeting of the Manchester branch of the Mathematical Association of England.

An informal colloquium on theory of numbers was held at Bristol, England on June 11-15. The visitors were guests of the mathematics department of the University of Bristol.

G. E. Stechert and Co., 31 East 10th St., New York City sends on request its catalog number 76, which contains a list of over fifteen hundred mathematical books and periodicals which it sells. The supply of periodicals includes bound and unbound sets of ninety-five different mathematical journals. Books listed include many rare and out-of-print ones, as well as the most recent treatises and texts. This catalog would be a valuable aid to one who is expanding his personal or departmental library. There is also a supplementary list of second hand volumes.

Professor G. F. McEwen is conducting experiments concerned with turbulence in the ocean at the Scripps Institution of Oceanography, La Jolla, California. Professor McEwen is a regular contributor to the meetings of the Southern California section of the American Mathematical Association.

L. J. ADAMS.

Problem Department

Edited by

T. A. BICKERSTAFF

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

While it is our aim to publish problems of most interest to the readers, it is believed that regular text-book problems are, as a rule, less interesting than others. Therefore, other problems will be given preference when the space for problems is limited.

Send all communications about problems to T. A Bickerstaff, University, Mississippi.

SOLUTIONS

No. 23. Proposed by William E. Byrne, Virginia Military Institute, Lexington, Virginia.

Find the most general solution of the simultaneous equations,

$$\cos\Theta = \frac{1+\cos\alpha}{|2\cos\alpha/2|}, \quad \sin\Theta = \frac{\sin\alpha}{|2\cos\alpha/2|}$$

where $|\quad|$ indicates absolute value and α is a constant, $\cos\alpha/2 \neq 0$.

Solution by H. T. R. Aude, Colgate College.

Dividing the second equation by the first and using a simple trigonometric reduction yields

$$\tan\Theta = \frac{\sin\alpha}{1+\cos\alpha} = \tan(\alpha/2)$$

The general solution of equation (1) is (1)

$$\Theta = n\cdot\pi + \alpha/2 \qquad (n=0,\ \pm1,\ \pm2,\ldots.) \tag{2}$$

But examination of the two given equations shows that the values of Θ given in (2) cannot hold for any values of the constant α since

cos Θ is never negative. It is thus seen that if cos $\alpha/2 > 0$, then n must must be an even number, and if cos $\alpha/2 < 0$, then n can only be an odd number. The most general solution is therefore dependent upon the size of the given angle α and can be written as follows:

If $\cos(\alpha/2) > 0$, then $\Theta = n \cdot \pi + \alpha/2$, $(n = 0, \pm 2, \pm 4, \ldots)$;

if $\cos(\alpha/2) < 0$, then $\Theta = n \cdot \pi + \alpha/2$, $(n = \pm 1, \pm 3, \ldots)$.

No. 24. Proposed by William E. Byrne, Virginia Military Institute, Lexington, Virginia.

Prove that if K is a given positive integer, the inequalities

$$\frac{(s-1)(s-2)}{2} < K \ngtr \frac{s(s-1)}{2}$$

Admit one and only one positive integral solution s. Find this solution for $K = 1000$.

Solution by H. T. R. Aude, Colgate College.

For K any positive integer it is evident that the value of $\sqrt{2K+\frac{1}{4}}$ is near a certain integer p such that the following relation holds

$$p - \tfrac{1}{2} \leqq \sqrt{2K+\tfrac{1}{4}} \ngtr p + \tfrac{1}{2}$$

Upon squaring these expressions and simplifying there results the relationship

$$\frac{p(p-1)}{2} \leqq K \ngtr \frac{p(p+1)}{2}$$

Next, replace p by $s-1$ and the proof is complete. Note, that in the problem as stated the equality sign should be added on the left side of K as in the proof above.

If $K = 1000$, then $\sqrt{2K+\frac{1}{4}}$ is between $45-\frac{1}{2}$ and $45+\frac{1}{2}$, hence $s = 46$, and

$$\frac{45 \cdot 44}{2} < 1000 \ngtr \frac{45 \cdot 46}{2}$$

But, if $K = 990$, then the equality sign must be written on the left side.

No. 28. Proposed by H. T. R. Aude, Colgate College.

Find the area in the xy-plane enclosed by the axes and

$$\cos^{-1}(x+y)-x-y=0.$$

Solution by the proposer.

Replacing $x+y$ by u allows the given equation to be written

$$u=\cos u.$$

This equation has only one real solution, namely $u=.73908$. The graph of the given equation is therefore that of the straight line $x+y=.73908$. The area enclosed by this line and the axes is $A=.27312$.

No. 63. Proposed by Walter B. Clarke, San Jose, California.

Line AB is one side of a triangle. Vertex C is to be located so that the projection on AB of HO equals that of IV. Point H is the orthocenter, O is the circumcenter, I is the incenter, and V is the concurrent point of three lines, each drawn from a vertex to the point half way around the perimeter of the triangle. What is the locus of C?

Solution by the proposer.

Assume $\qquad a<b<c$

The distance from B to the foot of the perpendicular from V to AB equals

$$(1) \qquad \frac{(c-a)^2+b(2c-b)}{2c}$$

Distance from B to the foot of the perpendicular from I to AB equals

$$(2) \qquad p-b=\frac{a-b+c}{2}=\frac{ac-bc+c^2}{2c}$$

Hence the projection of IV on AB is (1) - (2) equals

$$(3) \qquad \frac{(b-a)(3c-a-b)}{2c}$$

The projection of HO on AB is

$$(4)\qquad \frac{c}{2} - \frac{a^2-b^2+c^2}{2c} = \frac{b^2-a^2}{2c}$$

Now when (3) equals (4),

$$3c-a-b=a+b$$

or

$$c=\tfrac{2}{3}(a+b)$$

This relation can be satisfied only by an ellipse with eccentricity $\frac{2}{3}$.

PROBLEMS FOR SOLUTION

No. 104. Proposed by Walter B. Clarke, San Jose, Cal.

Three lines, a, b, and c, which are neither concurrent nor parallel intersect a fourth line XY in the points A, B, and C respectively. Lines a and b intersect at D; b and c intersect at E; and a and c intersect at F. On ED, take EG = CE and GH = CF (EH > EG). Through G, draw a parallel to FH cutting FC at K. Show that FB, AE, and KD, are concurrent regardless of which way EG is taken.

No. 105. Proposed by Raymond A. Lyttleton, Princeton University.

If an ellipse is inscribed in a triangle with its center at the circumcenter, then the altitudes are normals.

No. 106. Proposed by Walter B. Clarke.

Using the notation: G for centroid, K for Nagel point, I for incenter, and H for orthocenter, prolong IG to M so that GM = $\frac{1}{2}$ IG. Show that H, M, and K are collinear and that HM = MK.

No. 107. Proposed by William E. Byrne, Virginia Military Institute.

The integral, $\displaystyle\int_0^{+\infty} \frac{e^{-x}dx}{\sqrt{x}\log(x+1)}$ is divergent.

Find a function, $\Theta(x)$ such that

$$\int_0^{+\infty} \left[\frac{e^{-x}}{\sqrt{x}\log(x+1)} - \Theta(x) \right] dx$$

is convergent.

No. 108. Proposed by Walter B. Clarke.

Using notation, G for centroid, H for orthocenter, I for incenter, O for circumcenter, and V for verbicenter, (which is the concurrent of lines from each vertex to the point half way around the perimeter. Show that HIG and GOV are equal in area.

No. 109. Integrate

$$\int_0^a \frac{1}{y^2}\left(\frac{\sqrt{2a^2-y^2}}{a\sqrt{2}} - \frac{a}{\sqrt{a^2+y^2}}\right) dy$$

assuming that a is positive.

LATE SOLUTIONS

No. 95. By John H. Smith, Cedar Falls, Iowa.

No. 84. By A. C. Briggs, Wilmington, Ohio.

HISTORY OF MATHEMATICS

The history of mathematics, of which there are a number of authoritative works, by no means difficult reading for a teacher of mathematics, throws much light upon the growth of the various branches of the science, shows much of the human side of this very abstract type of human thinking and explains much of the gradual civilization of the human race. This history is fascinating, a source of understanding and interpretation if read and reread at intervals from year to year. The great figures in mathematics, many of them intensely human, clothe the science in flesh and blood, for our youth. A familiarity with the anecdotes associated with these men, with the account of the development of mathematics out of very genuine human every-day needs, and with the spread of mathematics over the world vitalizes the subject, making more of it than mere abstractions, ready made. After all, it will possibly be granted that the gradual development by the human race of a flexible system of numbers so that it, in time, could count has as largely influenced civilization as the invention of the alphabet.—Laura Blank in *Mathematics Teacher*, November, 1930.

Book Reviews

Edited by
P. K. SMITH

Financial Mathematics. By A. W. Richeson, New York, Prentice-Hall, Inc., 1935. vii+361 pp. $2.50.

In the preface, the author states the text is intended primarily for students majoring in business administration, but that it is designed so that it may be used with profit by those who are not specializing in that subject. The book is divided into four parts: interest and annuities, probability and life insurance, review of subjects from algebra, and 15 tables.

The book is well written in simple language. It appears to be a happy medium between the theoretical and the practical side of the subject. Brief introductory discussions of the various topics are given before the theory is taken up. The explanations are careful and complete; derivations of the formulas, logical and clear-cut; and definitions, concise. The text is not "cluttered up" with a lot of useless formulas and illustrated exercises.

The selection of the problems seems very good. Students will not form the idea that all rates are 4%!. The range of interest rates is somewhat greater than appears in similar texts. Review exercises are given at the end of each chapter, and at the conclusion of Part II, 159 practical miscellaneous problems are compiled.

For a first edition there are perhaps a minimum of errors. The book seems to be very usable.

HENRY A. ROBINSON.

Elementary Algebra. By E. I. Edgerton of W. L. Dickson High School, Jersey City; N. J.; and R. A. Carpenter of West High School, Rcchester, N. Y. XLI—498 pages. 57 illustrations, Allyn & Bacon 1934.

This book is designed for pupils beginning the study of algebra. It is divided into three parts. Part I starting with simple algebraic notations and formulas goes through simple equations. There are 182 pages devoted to part I. Part II beginning with products and factors carries the student through quadratic equations. There are 226 pages devoted to part II. Part III covering 46 pages includes ratio, proportion, variation, dependence, and numerical trigonometry.

In addition to the main body of the book there are the introduction including 33 pages of statistical graphs serving as a bridge between arithmetic and algebra, and 44 pages of supplementary exercises and examinations in the back of the book.

Tests of all kinds and accumulative reviews abound throughout the book and at the end of each chapter there is an equation review. The book contains an abundance of good examples and problems. The illustrations are attractive and should add life to the study of algebra.

Teachers of high school algebra will find this book interesting.

HENRY SCHROEDER.

The Calculus. Third edition. By Dalaker and Hartig. McGraw Hill, New York, 1935, viii+277 pp.

A brief preface indicates that the purpose of the text is to develop an introductory course. In giving an appraisal of such a book, observations may ordinarily fall in two groups: first, subjects and amount of development which would appear in any text; second, arrangement, and presentation of topics whereby opinions and needs may differ. The first group would certainly include the following: Statement of Introductory Notions, The Derivative, Application to Algebraic and Transcendental functions, Applications to curves and motions, Theorem of the Mean, Approximations, Partial derivative, General Integral, Definite Integral, Multiple Integral, Series and Expansion, Geometric and Physical Applications of the Integral, Problem lists. A brief and concise treatment of each of these topics appears in the text. A chapter on ordinary differential equations is introduced without applications, to problems. The last chapter gives a brief list of formulas, curves, and integrals.

Referring to the second type of observation we note that developments in the text proceed as follows: Introductory notions, Derivatives of standard types of functions with lists of equations for practice, applications to various types of problems beginning on page 57 and continuing through page 98. Integration is developed in a similar fashion. According to this plan the mathematics is first developed and then a variety of applications to applied fields follow. This approach differs from the plan of introducing the applied problem in connection with the process which applies. Some hold that this somewhat objective approach is preferable for the engineer and the student of applied science. See in this connection a paper by Dudley, *American Mathematical Monthly,* May, 1935. Other points on which opinions may differ include distribution of practice materials, and details in connection with the so-called non-rigorous proof. This

text is well in line with other popular texts on these points, and the illustrative drawings and general appearance of the book are very good.

We have attempted to indicate in a general way what the book contains and what the prominent features are without presuming to say who may or may not prefer to use it. In fact, a statement of what we may or may not want in a calculus would properly belong in a different paper altogether. C. D. SMITH.

THE PUBLIC AND MATHEMATICS

The best thing to do is to devote a little time and effort to pointing out the essential unsoundness of the arguments that have been advanced against the retention of mathematics as a required subject in the high schools. The movement against mathematics is, for the most part, confined to a group of educational theorists who feel that they must advocate something new in order to convince their readers that they are investigators. This group, however, has made up in volume of sound for what it has lacked in numbers, and in consequence, has deceived many people into thinking that it represents a widespread trend of thought.

The statement that there is at the present time much uncertainty as to the educational value of algebra and geometry will not bear examination. That the thinking public is as firmly convinced as ever of the high educational value of these subjects was conclusively shown by the answers to a questionnaire sent out recently by Professor Hancock. This questionnaire was sent to a group of the most prominent physicians, clergymen, lawyers and business men throughout the country; it was also sent to a similar group of residents of Cincinnati. In the first group 90 out of a total of 99 advocated a high school course in which mathematics was a required subject; in the second group 96 out of a total of 105 did the same thing. In view of these replies it seems to the writer that the burden of proof rests upon those who wish to displace mathematics as a required subject for high school pupils.

The statment that recent investigations have thrown doubt on the disciplinary value of mathematical study is absolutely without justification. The chief disciplinary value of mathematics is in the training of the reasoning powers it affords. The writer has gone over the literature on transfer of training quite recently and does not know of any experiments that involved training in logical reasoning. It is absurd to contend that experiments based on the marking of certain letters on a printed page or guessing the size of pieces of paper, will enable one to draw valid conclusions with regard to the training afforded by the study of mathematics.—Charles N. Moore in ***American Mathematical Monthly***, February, 1916.